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Threel string junction and $\mathcal{N} = 4$ dyon spectrum

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Abstract: The exact spectrum of dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories gives us information about dyon spectrum in $\mathcal{N} = 4$ supersymmetric gauge theories. This in turn can be translated into prediction about the BPS spectrum of three string junctions on a configuration of three parallel D3-branes. We show that this prediction agrees with the known spectrum of three string junction in different domains in the moduli space separated by walls of marginal stability.

Keywords: Superstrings and Heterotic Strings, D-branes.

We now have a good understanding of the exact spectrum of a class of quarter BPS dyons in a variety of $\mathcal{N} = 4$ supersymmetric string theories [1 − 18]. Since by going to the appropriate region in the moduli space of these string theories and taking a decoupling limit we can recover $\mathcal{N} = 4$ supersymmetric gauge theories [19, 20], the spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories provides us information about the spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric gauge theories. The latter, in turn, can be related to the BPS spectrum of string junctions on a set of parallel D3 branes [21]. Thus the known dyon spectrum in $\mathcal{N} = 4$ supersymmetric string theories gives us prediction about the BPS spectrum of string junctions on a set of parallel D3-branes. Our goal is to verify if this prediction is consistent with the known properties of string junctions.

We shall work with heterotic string theory on $T^4 \times T^2$ and focus on the four U(1) gauge fields associated with the components of the metric and 2-form field along the T^2 directions. The electric charges associated with these gauge fields are the momenta $\widehat{n}$ and n' and the fundamental string winding charges $-\widehat{w}$ and $-w'$ along the two circles of T^2, and the magnetic charges associated with these gauge fields are the H-monopole charges $-\widehat{W}$ and $-W'$ and the Kaluza-Klein monopole charges $\widehat{N}$ and N' along the same two circles. Following the notations and conventions of [18] we define the electric and magnetic charge vectors as:

$$Q = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix}. \tag{1}$$

The complex structure and the (complexified) Kahler moduli of the torus T^2 are encoded in a 4×4 matrix M satisfying

$$M^T L M = L, \quad M^T = M\,, \tag{2}$$

where

$$L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3}$$

I_k is the $k \times k$ identity matrix. The other complex modulus relevant for our discussion is the axion-dilaton modulus $\tau = a + iS$, where a is the field obtained by dualizing the 2-form field in four dimensions and $S = e^{-2\phi}$, ϕ being the dilaton field.

The T-duality transformations associated with T^2 are generated by 4×4 matrices Ω with integer entries and satisfying $\Omega L \Omega^T = L$. They act on the charges and the moduli as

$$Q \to (\Omega^T)^{-1} Q, \quad P \to (\Omega^T)^{-1} P, \quad M \to \Omega M \Omega^T, \quad \tau \to \tau\,. \tag{4}$$

Thus the combinations

$$Q^2 = Q^T L Q, \quad P^2 = P^T L P, \quad Q \cdot P = Q^T L P \tag{5}$$

are T-duality invariant. On the other hand the S-duality transformations are generated by SL(2, $\mathbb{Z}$) matrices $\begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix}$ with $\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d} \in \mathbb{Z}$, $\widehat{a}\widehat{d} - \widehat{b}\widehat{c} = 1$, and act on the charges and the

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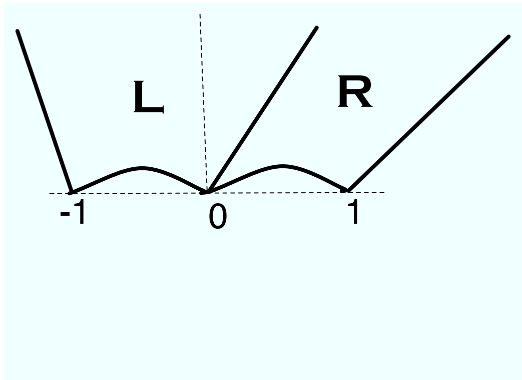

Figure 1: The domains $\mathcal{R}$ and $\mathcal{L}$.

moduli as

$$Q \to \widehat{a}Q + \widehat{b}P, \quad P \to \widehat{c}Q + \widehat{d}P\,, \quad \tau \to \frac{\widehat{a}\tau + \widehat{b}}{\widehat{c}\tau + \widehat{d}}, \quad M \to M\,. \tag{6}$$

As was reviewed in [18], for all charge vectors (Q, P) which are related to the charge vectors

$$Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} l_3 \\ l_4 \\ l_5 \\ 0 \end{pmatrix}, \qquad k_i, l_i \in \mathbb{Z}\,, \qquad \text{g.c.d.}(l_3, l_5) = 1\,, \tag{7}$$

by a T-duality transformation, we have a simple formula for the degeneracy (more precisely the number of bosonic supermultiplets minus the number of fermionic supermultiplets) of quarter BPS states:

$$d(Q, P) = (-1)^{Q\cdot P+1}\, g\left(\frac{1}{2}Q^2, \frac{1}{2}P^2, Q\cdot P\right)\,, \tag{8}$$

where $g(m, n, p)$ are defined as the coefficients of Fourier expansion of a known function $1/\widetilde{\Phi}$:

$$\frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})} = \sum_{\substack{m,n,p\in\mathbb{Z} \\ m\geq -1, n\geq -1}} g(m, n, p)\, e^{2\pi i(m\widetilde{\rho} + n\widetilde{\sigma} + p\widetilde{v})}\,. \tag{9}$$

In the particular theory under consideration, $\widetilde{\Phi}$ is the well known Igusa cusp form of weight 10 [22 − 25] on the moduli space of genus two Riemann surfaces, parametrized by the period matrix $\begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix}$ [1]. From (8), (9) we see that $d(Q, P) = 0$ unless $Q^2 \geq -2$ and $P^2 \geq -2$.

The formula for the degeneracy given above is not complete unless we specify the region of the moduli space in which the formula is valid. As we vary the asymptotic moduli the degeneracy can actually jump across walls of marginal stability, — codimension one subspaces of the moduli space on which the original quarter BPS dyon can decay into a pair of half-BPS dyons [13, 14, 16, 17, 26, 27]. As has been reviewed in detail in [18], a very useful way to label a given wall of marginal stability is to specify the relation between the charges of the decay products and the charges of the original state. In particular the possible decays of a quarter BPS state with charge (Q, P) are into half BPS states carrying charges $(adQ - abP, cdQ - cbP)$ and $(-bcQ + abP, -cdQ + adP)$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. For fixed values of the moduli M, the corresponding wall is either a circle in the τ plane intersecting the real axis at a/c and b/d, or – for $c = 0$ or $d = 0$ – a straight line passing through b/d or a/c. The radii of the circles and the slopes of the straight lines depend on Q, P and the other moduli M. The degeneracy formula given in (8), (9) is valid inside two separate domains bounded by walls of marginal stability. The first domain, called $\mathcal{R}$, is bounded by three different walls, — a straight line through 0, a circle connecting 0 and 1 and a straight line through 1 (see figure 1). From the decay rules given above it is clear that these three domain walls correspond to the decays:

$$(Q,P) \to (Q,0)+(0,P), \quad (Q,P) \to (0,-Q+P)+(Q,Q), \quad (Q,P) \to (P,P)+(Q-P,0)\,. \tag{10}$$

The other domain $\mathcal{L}$ inside which the degeneracy formula is valid is bounded by a straight line through 0, a circle through -1 to 0 and a straight line through -1. By following the same rules we see that these walls correspond to the possible decays:

$$(Q,P) \to (Q,0)+(0,P), \quad (Q,P) \to (0,Q+P)+(Q,-Q), \quad (Q,P) \to (-P,P)+(Q+P,0)\,. \tag{11}$$

Even though the degeneracy formula (8) hold in both domains $\mathcal{R}$ and $\mathcal{L}$, there is a subtle difference between the ways we extract $g(m, n, p)$ from (9) in the two cases. When we are computing the formula in the domain $\mathcal{R}$, we need to expand $1/\widetilde{\Phi}$ in such a way that for a fixed m, n, the sum over p is bounded from above. On the other hand inside the domain $\mathcal{L}$ we have to expand $1/\widetilde{\Phi}$ so that for fixed m, n the sum over p is bounded from below.

We shall first focus on the degeneracy of states with $Q^2 = P^2 = -2$ and later consider states related to these by S-duality transformation. For this we only need to examine terms in $1/\widetilde{\Phi}$ whose $\widetilde{\sigma}$, $\widetilde{\rho}$ dependence is of the form $e^{-2\pi i\widetilde{\rho}-2\pi i\widetilde{\sigma}}$. The relevant part of $1/\widetilde{\Phi}$ is

$$\frac{1}{\widetilde{\Phi}} \simeq e^{-2\pi i\widetilde{\rho}-2\pi i\widetilde{\sigma}} \frac{e^{-2\pi i\widetilde{v}}}{\left(1-e^{-2\pi i\widetilde{v}}\right)^2}\,. \tag{12}$$

According to the rule given above we need to expand this in power of $e^{-2\pi i\widetilde{v}}$ for calculating the degeneracy in the domain $\mathcal{R}$. This gives, in the domain $\mathcal{R}$,

$$d(Q,P) = \begin{cases} 0 \quad \text{for } Q^2 = P^2 = -2,\ Q\cdot P \geq 0 \\ j(-1)^{j-1} \quad \text{for } Q^2 = P^2 = -2,\ Q\cdot P = -j,\ j>0 \end{cases}. \tag{13}$$

On the other hand in the domain $\mathcal{L}$ we have to expand $1/\widetilde{\Phi}$ in powers of $e^{2\pi i\widetilde{v}}$ by expressing the $\widetilde{v}$ dependent factor as $e^{2\pi i\widetilde{v}}/(1-e^{2\pi i\widetilde{v}})^2$. This gives, in the domain $\mathcal{L}$,

$$d(Q,P) = \begin{cases} 0 \quad \text{for } Q^2 = P^2 = -2,\ Q\cdot P \leq 0 \\ j(-1)^{j+1} \quad \text{for } Q^2 = P^2 = -2,\ Q\cdot P = j,\ j>0 \end{cases}. \tag{14}$$

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Let us now focus on a particular state carrying charge vectors

$$Q_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{15}$$

This is of the form given in (7), and has

$$Q_0^2 = -2, \qquad P_0^2 = -2, \qquad Q_0 \cdot P_0 = -1\,. \tag{16}$$

Thus according to (13), (14), this state will have degeneracy 1 in the domain $\mathcal{R}$ and vanishing degeneracy in the domain $\mathcal{L}$. In other words, the state will cease to exist as we move from the domain $\mathcal{R}$ to the domain $\mathcal{L}$ crossing the wall separating the two domains. In the τ plane this wall is a straight line through 0.

We shall now examine the fate of the state in various other domains. This is done by noting that the degeneracies in the other domains may be calculated by mapping them to the domain $\mathcal{R}$ using an S-duality transformation and then applying the degeneracy formula (8), (9) in the domain $\mathcal{R}$ [13]. Let us consider a domain $\widetilde{\mathcal{R}}$ that is mapped to the domain $\mathcal{R}$ via an S-duality transformation matrix $\begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix}$. This will map the charge vector (Q_0, P_0) to (Q_0', P_0') given by

$$Q_0' = \widehat{a} Q_0 + \widehat{b} P_0 = \begin{pmatrix} -\widehat{b} \\ \widehat{a}+\widehat{b} \\ \widehat{b} \\ -\widehat{a} \end{pmatrix}, \qquad P_0' = \widehat{c} Q_0 + \widehat{d} P_0 = \begin{pmatrix} -\widehat{d} \\ \widehat{c}+\widehat{d} \\ \widehat{d} \\ -\widehat{c} \end{pmatrix}. \tag{17}$$

Thus $d(Q_0, P_0)$ in the domain $\widetilde{\mathcal{R}}$ is equal to $d(Q_0', P_0')$ in the domain $\mathcal{R}$. Although the charge vectors (Q_0', P_0') do not have the form given in (7), they can be expressed as

$$Q_0' = (\Omega^T)^{-1} Q_0'', \qquad P_0' = (\Omega^T)^{-1} P_0''\,, \tag{18}$$

where

$$Q_0'' = \begin{pmatrix} 0 \\ \widehat{a}^2 + \widehat{a}\widehat{b} + \widehat{b}^2 \\ \widehat{a}\widehat{c} + \widehat{b}\widehat{c} + \widehat{b}\widehat{d} \\ -1 \end{pmatrix}, \qquad P_0'' = \begin{pmatrix} -1 \\ \widehat{a}\widehat{c} + \widehat{a}\widehat{d} + \widehat{b}\widehat{d} \\ \widehat{c}^2 + \widehat{c}\widehat{d} + \widehat{d}^2 \\ 0 \end{pmatrix}, \qquad \Omega^T = \begin{pmatrix} \widehat{a} & 0 & 0 & -\widehat{b} \\ 0 & \widehat{a} & \widehat{b} & 0 \\ 0 & \widehat{c} & \widehat{d} & 0 \\ -\widehat{c} & 0 & 0 & \widehat{d} \end{pmatrix}. \tag{19}$$

Since $(Q_0'',\, P_0'')$ have the form (7), and Ω denotes a T-duality transformation, we conclude that our degeneracy formula (8), (9) holds for the charge vectors (17). Thus in order to get non-vanishing $d(Q_0', P_0')$ we must have $(Q_0')^2 \geq -2$, $(P_0')^2 \geq -2$. Using (17) these conditions translate to

$$\widehat{a}^2 + \widehat{b}^2 + \widehat{a}\widehat{b} \leq 1, \qquad \widehat{c}^2 + \widehat{d}^2 + \widehat{c}\widehat{d} \leq 1\,. \tag{20}$$

Since the left hand sides of both equations are positive definite for $\widehat{a}\widehat{d} - \widehat{b}\widehat{c} = 1$, the above equations give strong constraint on $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$. In particular for integer $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{d}$ both bounds must be saturated. Thus we have $(Q_0')^2 = (P_0')^2 = -2$. Eq.(13) now tells us that unless $Q_0' \cdot P_0' \leq -1$ the degeneracy vanishes in the domain $\mathcal{R}$. This gives rise to one more inequality

$$2(\widehat{a}\widehat{c} + \widehat{b}\widehat{d}) + \widehat{a}\widehat{d} + \widehat{b}\widehat{c} \geq 1\,. \tag{21}$$

We can find all integer solutions to (20), (21) subject to the restriction $\widehat{a}\widehat{d} - \widehat{b}\widehat{c} = 1$. Up to an overall sign that does not affect the mapping between the domains in the τ plane, we get the following solutions for $\begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix}$:

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{22}$$

This gives the set of all $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$ for which $d(Q_0', P_0')$ is non-zero inside $\mathcal{R}$ and hence $d(Q_0, P_0)$ is non-zero inside $\widetilde{\mathcal{R}}$. Thus the set of all domains in which $d(Q_0, P_0)$ is non-zero is obtained by the image of $\mathcal{R}$ (for charge vector (Q_0', P_0')) under an S-duality transformation by the inverse of $\begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix}$. Now one can easily verify that each of the S-duality transformations given in (22) maps the domain $\mathcal{R}$ to itself,[1] – this is best seen by noting that each of these transformations permutes the vertices 0, 1 and ∞ of $\mathcal{R}$. Thus any other domain $\widetilde{\mathcal{R}}$ is mapped to $\mathcal{R}$ via an S-duality transformation outside the set (22), and hence $d(Q_0, P_0)$ must vanish in the domain $\widetilde{\mathcal{R}}$. This leads to the conclusion that $d(Q_0, P_0)$ vanishes in all domains outside $\mathcal{R}$.

We shall now try to verify this prediction by working near a point in the moduli space where there is an enhanced SU(3) gauge symmetry. This is achieved by taking the matrix valued scalar field M to be of the form

$$M = \begin{pmatrix} 4/3 & 2/3 & -1/3 & 2/3 \\ 2/3 & 4/3 & -2/3 & 1/3 \\ -1/3 & -2/3 & 4/3 & -2/3 \\ 2/3 & 1/3 & -2/3 & 4/3 \end{pmatrix}. \tag{23}$$

In this case one finds that the BPS mass of a purely electric state vanishes for the charge vectors

$$\alpha = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}. \tag{24}$$

Indeed the vectors $\pm\alpha$, $\pm\beta$ and $\pm\gamma = \pm(\alpha+\beta)$ are eigenvectors of $(M+L)$ with zero eigenvalue and satisfy $\alpha^2 = \beta^2 = \gamma^2 = -1$. As a result electrically charged states with these charge vectors give the six massless electrically charged states which are necessary for getting the full set of SU(3) gauge fields. If we adjust the moduli M to be slightly away from the one given in (23) then the SU(3) gauge symmetry is spontaneously broken

[1] More precisely, it maps the domain $\mathcal{R}$ for the charge vector (Q_0, P_0) to the domain $\mathcal{R}$ for the charge vector (Q_0', P_0').

to $\mathrm{U}(1) \times \mathrm{U}(1)$ and the charged gauge fields become massive, but remain light compared to the string scale.

Besides the half-BPS massive gauge fields, and the half-BPS dyons related to these by S-duality transformation, the spontaneously broken SU(3) gauge theory also contains quarter BPS dyons carrying electric and magnetic charges of the form [21]

$$Q = p\alpha + q\beta, \quad P = r\alpha + s\beta, \quad p, q, r, s \in \mathbb{Z}\,, \tag{25}$$

in specific domains in the moduli space depending on the values of p, q, r, s. These must represent some states in the spectrum of quarter BPS dyons in string theory.[2] Conversely every BPS state in string theory, carrying charge vectors of the form $(p\alpha + q\beta, r\alpha + s\beta)$, becomes light compared to the string scale in the region of the moduli space we are considering, and hence they must have a realization in gauge theory. In particular since (Q_0, P_0) defined in (15) has the form

$$Q_0 = \beta, \qquad P_0 = -\alpha\,, \tag{26}$$

it must have a realization in SU(3) gauge theory.

A simple realization of quarter BPS dyons in SU(3) gauge theory is as a 3-string junction [28, 29] on a system of three closeby parallel D3-branes [21]. This system has $\mathrm{U}(1)\times$ spontaneously broken SU(3) gauge theory as its low energy limit. An (m, n) string ending on a D3-brane carries an electric charge m and magnetic charge n under the U(1) gauge field living on the D3-brane. Thus if we have a configuration where an (m_1, n_1) string ends on the first D3-brane, an (m_2, n_2) string ends on the second D3-brane and an (m_3, n_3) string ends on the third D3-brane, with the three strings meeting at a 3-point junction, then the system will be said to carry electric and magnetic charge vectors

$$\widetilde{Q} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \widetilde{P} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{27}$$

Charge conservation at the three string junction requires $\sum_i m_i$ and $\sum_i n_i$ to vanish. Thus although the gauge theory on the D3-brane is U(3), the state described above carries only an SU(3) charge. This allows us to compare a BPS state of the configuration described above with that of the SU(3) gauge theory that arises as the low energy limit of heteroric string theory on $T^4 \times T^2$. For this we first need to learn how to translate a charge vector of the type given in (27) to the one given in (1). We do this by comparing the charges carried by the massive gauge fields. On the configuration of three D3-branes, the massive gauge fields arise from (1,0) string stretching from one D3-brane to another. Thus in the convention described above, the electric charges carried by these gauge fields take the form $\pm\widetilde{\alpha}$, $\pm\widetilde{\beta}$ and $\pm\widetilde{\gamma} = \pm(\widetilde{\alpha} + \widetilde{\beta})$ with

$$\widetilde{\alpha} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \widetilde{\beta} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \widetilde{\gamma} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{28}$$

[2] Note that the gauge theory limit can be taken by adjusting the moduli M and is insensitive to $\tau = a+iS$. Thus even after taking the gauge theory limit we can explore all the different domains separated by walls of marginal stability by varying τ.

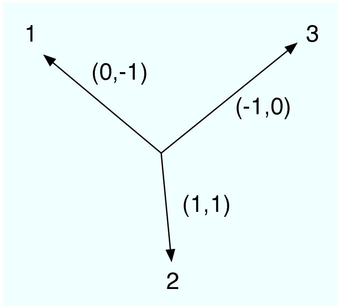

Figure 2: The three string junction containing a $(0,-1)$ string, a (1,1) string and a $(-1,0)$ string ending on D3 branes 1, 2 and 3 respectively.

On the other hand in heterotic string theory on $T^4 \times T^2$, the SU(3) gauge fields carry electric charge vectors $\pm\alpha$, $\pm\beta$ and $\pm\gamma$ given in (24). Thus we now have a correspondence between the charge vectors in the D3-brane system to ones in heterotic string theory for states which are charged only under the SU(3) subgroups in both theories. In particular a state in the heterotic string theory carrying charges $(p\alpha + q\beta, r\alpha + s\beta)$ will correspond to a three string junction carrying charges

$$\widetilde{Q} = p\widetilde{\alpha} + q\widetilde{\beta}, \qquad \widetilde{P} = r\widetilde{\alpha} + s\widetilde{\beta}\,. \tag{29}$$

Let us now consider a three string junction in which a $(0,-1)$ string ends on the first D3-brane, a $(1,1)$ string ends on the second D3-brane and a $(-1,0)$ string ends on the third D3-brane (see figure 2). This corresponds to the choice

$$\widetilde{Q} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \widetilde{\beta}, \qquad \widetilde{P} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -\widetilde{\alpha}\,. \tag{30}$$

Thus in heterotic string theory on $T^4 \times T^2$, the charge vectors carried by this state will be $(\beta, -\alpha) = (Q_0, P_0)$ with Q_0, P_0 given in (15).

We shall now examine the domain in which the three string junction exists and compare the result with the one obtained from the dyon degeneracy formula in $\mathcal{N} = 4$ supersymmetric string theory. The 3-string junction can become marginally unstable in one of three ways, corresponding to shrinking one of the three strings to zero size [21] (see figure 3). Consider first the case where the $(0,-1)$ string ending on the first D3-brane shrinks to zero size. In this case the resulting configuration becomes identical to that of a (1,1) string going from the first to the second brane and a $(-1,0)$ string going from the first to the third brane. According to our convention the former has charge vectors

$$\widetilde{Q}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \widetilde{P}, \quad \widetilde{P}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \widetilde{P}, \tag{31}$$

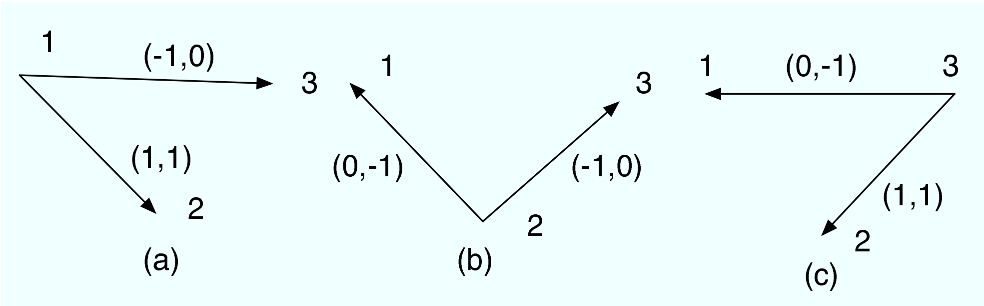

Figure 3: Marginally stable three string junctions. In (a) the string ending on the first D3 brane shrinks to zero size, in (b) the string ending on the second D3-brane shrinks to zero size, and in (c) the string ending on the third D3-brane shrinks to zero size.

whereas the latter has a charge vector

$$\widetilde{Q}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \widetilde{Q} - \widetilde{P}, \qquad \widetilde{P}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\,. \tag{32}$$

Thus this particular wall of marginal stability corresponds to the decay

$$(\widetilde{Q}, \widetilde{P}) \to (\widetilde{P}, \widetilde{P}) + (\widetilde{Q} - \widetilde{P}, 0)\,. \tag{33}$$

Proceeding this way we see that the second wall of marginal stability, corresponding to shrinking of the (1,1) string ending on the second D3-brane to zero size, induces decay into an $(0,-1)$ string going from the second to the first brane and a $(-1,0)$ string going from the second to the third branes. This gives

$$(\widetilde{Q}, \widetilde{P}) \to (0, \widetilde{P}) + (\widetilde{Q}, 0)\,. \tag{34}$$

Finally the third wall of marginal stability, corresponding to the shrinking of the $(-1,0)$ string ending on the third brane to zero size, induces decay into a $(0,-1)$ string going from the third to the first brane and a $(1,1)$ string going from the third to the second brane. This gives

$$(\widetilde{Q}, \widetilde{P}) \to (0, \widetilde{P} - \widetilde{Q}) + (\widetilde{Q}, \widetilde{Q})\,. \tag{35}$$

Eqs.(33), (34) and (35) give the walls of marginal stability bordering the domain in which the three string junction under consideration exists. From (10) we see that these are precisely the walls which border the domain $\mathcal{R}$. Thus we see that the particular three string junction under consideration exists in the domain $\mathcal{R}$, — exactly as predicted by the dyon degeneracy formula in string theory. The degeneracy formula in fact goes further and predicts that these states will have degeneracy 1. This cannot be verified directly using the three string junction picture since quantization of such a configuration is difficult, but has been verified by working in the gauge theory description of these states [30−32].

We can also consider a slightly different three string junction in which a $(0,1)$ string ends on the first D3-brane, a $(1,-1)$ string ends on the second D3-brane and a $(-1,0)$ string ends on the third D3-brane. Following the same procedure as in the previous case one finds that the corresponding state in heterotic string theory on $T^4 \times T^2$ has $Q^2 = P^2 = -2$, $Q \cdot P = 1$. Thus this state exists with degeneracy 1 in the domain $\mathcal{L}$. An analysis identical to the one described earlier shows that in the heterotic string theory description the state ceases to exist as we cross any of the walls of marginal stability bordering the domain $\mathcal{L}$. This leads to a definite prediction for the domain in the moduli space of D3 brane configurations in which the three string junction exists. This can be verified explicitly in the same way as in the previous case.

Let us now consider the case of a more general three string junction configuration where an (m_1, n_1) string ends on the first D3-brane, an (m_2, n_2) string ends on the second D3-brane and an (m_3, n_3) brane ends on the third D3-brane. If $(m_1 n_3 - m_3 n_1) = -1$ then this configuration may be obtained from the one discussed earlier, — containing a $(0, -1)$, $(1, 1)$ and $(-1, 0)$ strings, — by an S-duality transformation by the matrix $\begin{pmatrix} -m_3 & -m_1 \\ -n_3 & -n_1 \end{pmatrix}$. Thus the domain in which it exists can be determined, — both in the string junction description and in the description as a dyon in the $\mathcal{N} = 4$ supersymmetric heterotic string theory, — by an S-duality transformation of the domain in which the configuration of $(0, -1)$, $(1, 1)$ and $(-1, 0)$ string exists. Our earlier analysis for the latter configuration now implies that the results in the two descriptions would also agree for the more general configuration involving (m_1, n_1), (m_2, n_2) and (m_3, n_3) strings as long as $(m_1 n_3 - m_3 n_1) = -1$. Similarly if $(m_1 n_3 - m_3 n_1) = 1$ then we can relate this to the configuration of $(0, 1)$, $(1, -1)$ and $(-1, 0)$ string via an S-duality transformation $\begin{pmatrix} -m_3 & m_1 \\ -n_3 & n_1 \end{pmatrix}$, and the agreement between the results based on dyon spectrum in string theory and three string junction would follow as a consequence of a similar agreement for the $(0, 1)$, $(1, -1)$ and $(-1, 0)$ configuration.

What about the case when $(m_1 n_3 - m_3 n_1) \neq \pm 1$? It is instructive to see what kind of charge vectors the general (m_1, n_1), $(m_2, n_2) = (-m_1 - m_3, -n_1 - n_3)$, (m_3, n_3) string configurtion corresponds to in heterotic string theory. In the string junction description the charge vectors are

$$\widetilde{Q} = \begin{pmatrix} m_1 \\ -m_1 - m_3 \\ m_3 \end{pmatrix} = m_1 \widetilde{\alpha} - m_3 \widetilde{\beta}, \qquad \widetilde{P} = \begin{pmatrix} n_1 \\ -n_1 - n_3 \\ n_3 \end{pmatrix} = n_1 \widetilde{\alpha} - n_3 \widetilde{\beta}. \tag{36}$$

Thus in the heterotic string theory on $T^4 \times T^2$ this would correspond to the charge vectors

$$Q = m_1 \alpha - m_3 \beta = \begin{pmatrix} m_1 \\ -m_1 - m_3 \\ -m_1 \\ m_3 \end{pmatrix}, \qquad P = n_1 \alpha - n_3 \beta = \begin{pmatrix} n_1 \\ -n_1 - n_3 \\ -n_1 \\ n_3 \end{pmatrix}. \tag{37}$$

In this case it is easy to see that

$$\text{g.c.d.}(Q_i P_j - Q_j P_i; \quad i, j = 1, 2, 3, 4) = |m_1 n_3 - m_3 n_1| \,. \tag{38}$$

For $|m_1 n_3 - m_3 n_1| \neq 1$ these states are outside the duality orbit of the state (7) [33, 14, 18, 34]. Hence the currently known formula for degeneracy of dyons in $\mathcal{N} = 4$ supersymmtric string theories do not have any information about the spectrum of these states. Indeed, since one can now construct a non-premitive lattice vector from an integer linear combination of Q and P, even the structure of the marginal stability walls change, – in the decay (Q, P) into $(adQ - abP, cdQ - cbP)$ and $(-bcQ + abP, -cdQ + adP)$ the coefficient a, b, c, d need not all be integers any more.

Different aspects of the relation between dyon spectrum in supersymmetric gauge theories and string theories have been discussed in [35].

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